PERFECT HALVING, BICIMALS, AND A TOAD-IN-THE-HOLE

Jonny Griffiths and Mike Ollerton showcase their talents both culinary and mathematical

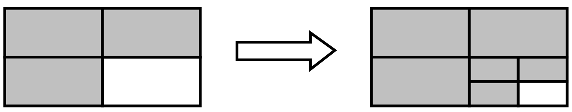
This writing arose from a sequence of events, which began at Conference 2011 conference in Telford with a conversation about '*bicimals*' between Jonny and Mike. This idea developed in October 2011 when Mike cooked *a toad in the hole* for three people and when, a couple of days later, Jonny sent Mike some ideas he had been working on with his Year12 group. The following story is not, therefore, entirely fictitious, though the *toad in the hole* could just as well have been made with vegetarian, or even Cumberland sausages…

Jonny:

Mike, that rectangular toad-in-the-hole you've cooked looks fantastic. But, how are you going to divide it up fairly? Andrew is coming around in a minute, so there will be three of us. Now I know that you are a perfect halver – you can halve any portion perfectly and quickly. But how are you going to use your halving skills to get three perfect thirds?

Mike:

Well that's easy Jonny. Being a perfect halver means I'm also a perfect quarterer, eighther, sixteenther, and so on. So I'll make four perfect quarters and we can have one piece each. Yes, I know what you're going to say next: what about the remaining quarter? Well, I am just going to repeat the same process again - see what I mean?



Jonny:

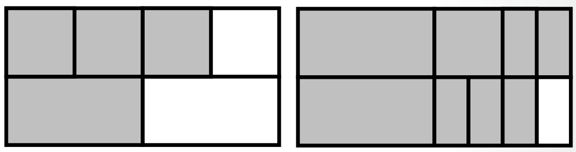
Ah, neat! So, Mike, we each get a $\frac{1}{4}$ *, then a quarter of a quarter, which is a* $\frac{1}{16}$ *, and then* $\frac{1}{64}$*th... and you are saying* $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$ *is a third? Well, the first term is* $\frac{1}{4}$*, and the number we multiply by each time in the sequence, the common ratio, is* $\frac{1}{4}$*.*

So, we have what I would call an infinite geometric series, and one with first term a and common ratio r has sum $\frac{a}{(1 - r)}$ *as long as the size of r is less than 1. In this case, the sum is* $(\frac{1}{4})/(1 - \frac{1}{4}) = \ldots \frac{1}{3}$*! It works, that's brilliant!*

However... I am sorry to put pressure on your halving skills, Mike, but I've just had a text to say two more friends are coming round, any second. That's five of us - how will your method produce perfect fifths?

Mike:

Jonny you really must try to be less profligate with your invitations. Just let me think... okay I've got it. We'll need to slice the toad into 8 equal pieces so we all get one each. That leaves 3 pieces which I can halve into 6 sixteenths so we can also have $\frac{1}{16}$ *each. This leaves a single sixteenth size piece, so we are back to the original problem of dividing one piece into five equal portions - you do the maths, Jonny...*



Jonny:

I just LOVE being told to do some maths! Okay, Mike, so this time you are saying we each get:

$\frac{1}{8} + \frac{1}{16} + (\frac{1}{8} \times \frac{1}{16}) + (\frac{1}{16} \times \frac{1}{16}) + (\frac{1}{8} \times \frac{1}{16} \times \frac{1}{16} \ldots)$ *or* $\frac{1}{8} + \frac{1}{16} + \frac{1}{128} + \frac{1}{256} + \frac{1}{2048} + \frac{1}{4096} \ldots$

Mmm... this looks like

$(\frac{1}{8} + \frac{1}{128} + \frac{1}{2048} \ldots) + (\frac{1}{16} + \frac{1}{256} + \frac{1}{4096} \ldots)$*, that is two infinite geometric series added together. So the total sum will be*

$\frac{1}{8}/(1 - \frac{1}{16}) + \frac{1}{16}/(1 - \frac{1}{16}) = \frac{2}{15} + \frac{1}{15} = \frac{1}{5}$.

It works, so forgive me, Mike, for temporarily doubting your halving abilities. But, can you get any fraction by halving?

… And all these powers of two are making me think in binary...

Mike:

Binary - now there's a powerful system. A little aside; on the 11th Nov 2011 or 11-11-11 I was 63 and it just so happens that 111111 in binary is equal to 63 in denary - good eh? Anyway, I'm also interested in 'bicimals' - they're like decimals *only in binary. So after 16, 8, 4, 2, 1 we have a bicimal point then we have $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ and so on. Now then if we return to the time when it was just you, me and Andrew who were going to eat the toad, as a bicimal this would have come out as 0.010101... or more succinctly 0.01 recurring. So, $\frac{1}{3}$ in denary is 0.01 recurring in binary. So what do you reckon $\frac{1}{5}$ would look like as a bicimal?*

Jonny:

Well, $\frac{1}{8} + \frac{1}{16} + \frac{1}{128} + \frac{1}{256} + \frac{1}{2048} + \frac{1}{4096}$... that gives 0.001100110011...? Another recurring bicimal, but then every fraction has to be recurring... Let me show you the same thing a different way. There are five of us. So Mike, you repeatedly halve the toad to give perfect sixteenths. We take three each, which leaves one piece – where you do the same again, and again... like you did with making thirds. So we each have $\frac{3}{16} + \frac{3}{(16^2)} + \frac{3}{(16^3)}$... an infinite geometric series with $a = \frac{3}{(16^3)}$, and $r = \frac{1}{(16^3)}$, so its sum is $(\frac{3}{16}) / (1 - \frac{1}{16}) = \frac{1}{5}$. Now, $\frac{3}{16}$ as a bicimal is 0.0011, while $\frac{3}{(16^2)}$ is 0.00000011, and so they add to give our recurring bicimal. Why does this work? Because 5 divides exactly into $2^4 - 1$, just as 3 divides exactly into $2^2 - 1$. So what happens with 7? Now, before you get worried, Mike, I haven't invited anyone else round without telling you...

Mike:

Phew, thank goodness for that. As for sevenths, this is a piece of cake which, by the way, I assume you are baking. Sevenths will be the same process as thirds, except we just need to slice the toad into eighths to begin with, then take one piece each thus leaving one piece and then we're back to the original situation. In binary this means $\frac{1}{7}$ is 0.001001001... or 0.001 all recurring. Putting this information into the formula we have $\frac{1}{8}/(1 - \frac{1}{8})$, which clearly computes to $\frac{1}{7}$.

Jonny:

...and this works because 7 divides into $2^3 - 1$. Now elevenths...

Mike:

These are a real test of one's patience...

Jonny:

Well, 11 divides into $2^{10} - 1$. So you halve the toad into 1024 equal pieces, and the eleven eaters take 93 pieces each, and we are left with one piece that we can repeat this on.

Mike:

Now 93 in binary is 1011101, which means that $\frac{1}{11}$ as a bicimal is...

Jonny:

0.0001011101000101110100010111 01...

Mike:

So that's a recurring bicimal of period 10 - that's good. Furthermore, once we know what $\frac{1}{3}$ and $\frac{1}{5}$ are as bicimals we can easily calculate $\frac{2}{3}$ and $\frac{2}{5}$ by shifting all the numbers one place to the left. So because $\frac{1}{3}$ = 0.01 then $\frac{2}{3}$ must be 0.10 recurring. Similarly because $\frac{1}{5}$ equals 0.0011 recurring then $\frac{2}{5}$ must be 0.0110 recurring. Also to find $\frac{3}{5}$ we just need to add $\frac{1}{5}$ and $\frac{2}{5}$ and in binary this is 0.1001 recurring.

Jonny:

Ah! I think that's a knock at the door. Did I mention that Andrew is a perfect trisector?

Mike:

Well that'll just lead us into tricimals… so for a start $\frac{1}{2}$ in base 3 is 0.1 recurring, perhaps we should ask Andrew what he thinks $\frac{1}{5}$ would be…

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